Linear System Theory – lecture 12

MIMO Polynomial Pole Placement

state-feedback

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Outline:

1. Column and row reducedness
2. MIMO polynomial pole placement
3. Example
4. State-feedback → realization
   Lyapunov equation
5. → Matlab experiment
6. What is the diff → pole assignment
   → eigenvalues placement.

1. Column and row reducedness

Def: d° of a polynomial vector

$\equiv$ highest power of s in all entries of the vector.

Def matrix $M(s)$

degree of the i_{th} column of $M(s)$

$$\equiv \delta_{ci}(M(s))$$

degree of the i^{th} row of $M(s)$

$$\equiv \delta_{ri}(M(s))$$

Example: $M(s) = \begin{bmatrix} s+1 & s^3 - 2s + 5 & -1 \\ s-1 & s^2 & 0 \end{bmatrix}$

$$\delta c_1 = 1, \quad \delta c_2 = 3, \quad \delta c_3 = 0$$

$$\delta r_1 = 3$$
$$\delta r_2 = 2$$

Def. A nonsingular $n \times n$ polynomial matrix $M(s)$ is column reduced if

$$\deg(\det(M(s))) = \sum_{i=1}^{n} \delta c_i$$

it is row reduced if

$$\deg(\det(M(s))) = \sum_{i=1}^{n} \delta r_i$$

Example:

$$M(s) = \begin{bmatrix} 3s^2 + 2s & 2s+1 \\ s^2 + s - 3 & s \end{bmatrix}$$

$$|M(s)| = s^3 - s^2 - 5s + 3 \qquad d^\circ(|M(s)|) = 3$$

$$d^\circ(M) = \sum_{i=1}^{2} \delta c_i = \delta c_1 + \delta c_2 = 2 + 1 = 3$$

$\Longleftrightarrow$ it is column reduced.

However, $M(s)$ is not row reduced

since $\delta r_1 + \delta r_2 = 2 + 2 = 4 > d^\circ(|M(s)|)$

$d^\circ(M)$

Notation:

$$\delta c_i(M(s)) = k_{ci}$$

$$H_c(s) = \text{diag}(s^{k_{c1}}, s^{k_{c2}}, \dots)$$

$$M(s) \triangleq \underbrace{M_{hc}}_{\text{no polynomials}} H_c(s) + M_{lc}(s)$$

(cf. S-matrix in the Structure theorem)

Example:

$$M(s) = \underbrace{\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}}_{M_{hc}} \begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} 2s & 1 \\ s-3 & 0 \end{bmatrix}$$

Definition: M_{hc} column-degree coefficient matrix.

Similar with rows, but M_{hr} will appear on the right

$$M(s) = H_r(s) M_{hr} + M_{lr}(s)$$

$$\delta r_i(M(s)) = k_{ri}$$

$$H_r(s) = \text{diag}(s^{k_{r1}}, s^{k_{r2}}, \dots)$$

Def: M_{hr} is called the row-degree coefficient matrix.

$M(s)$ is column reduced

$\Longleftrightarrow$ M_{hc} is nonsingular

Consider $$G(s) = N(s)\, D^{-1}(s)$$
$$= \bar{D}^{-1}(s)\, \bar{N}(s)$$

with $\begin{cases} N, D \text{ are right coprime} \\ \bar{N}, \bar{D} \text{ are left coprime} \\ D \text{ is column reduced and} \\ \bar{D} \text{ is row reduced} \end{cases}$

then we have

$$d^{\circ}(G(s)) = \sum \delta_{ci}(D(s))$$
$$= \sum \delta_{ri}(\bar{D}(s))$$

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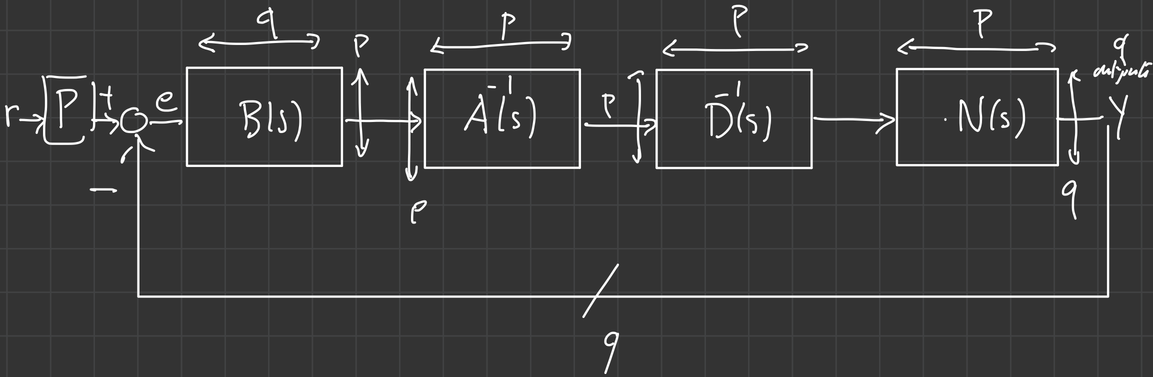
2. MIMO pole assignment

$$G(s) = N(s)\, D^{-1}(s)$$ (right polynomial fraction) coprime

q outputs

p inputs



$P = I_q$ $\qquad$ $A, D \quad p \times p$

$C(s) = A^{-1} B$ $\qquad$ $B \quad p \times q$

$N \quad q \times q$

$$y(s) = G(s)\, C(s)\, (r - y)$$

$$y + GCy = GCr$$

$$y \cdot r^{-1} = [I + GC]^{-1} GC$$

$$e(s) = r(s) - GCe$$

$$e = (I_d + GC)^{-1} r$$

$$y = GCe$$

$$GC\,(I_q + GC)^{-1}$$

$$u = C(r - Gu) \qquad G(s)\,[I_q + CG]^{-1} C$$

$$G(s) = N \cdot D^{-1}$$

$$C(s) = A^{-1} B$$

$$N D^{-1} \left[I + A^{-1} B N D^{-1} \right]^{-1} A^{-1} B$$

$$= N D^{-1} \left\{ A^{-1} \left[AD + BN \right] D^{-1} \right\}^{-1} A^{-1} B$$

$$= N \left[AD + BN \right]^{-1} B$$

$$= N \; F^{-1} \; B$$

$$\boxed{AD + BN = F}$$

$$\deg G(s) = \deg(\det(D(s)))$$

$$= \mu_1 + \mu_2 + \cdots + \mu_p = n$$

μ_i : degree of the i^{th} column of $D(s)$.

<u>Def:</u> $$\mu \triangleq \max(\mu_1, \mu_2 \dots, \mu_p)$$

$$p \updownarrow \overset{p}{\overleftrightarrow{D(s)}} = D_0 + D_1 s + \cdots + D_\mu s^\mu$$

$$q \updownarrow \overset{p}{\overleftrightarrow{N(s)}} = N_0 + N_1 s + \cdots + N_\mu s^\mu$$

Parameterization of the controller

$$A(s) = A_0 + A_1 s + A_2 s^2 + \cdots + A_m s^m$$

$$B(s) = B_0 + B_1 s + B_2 s^2 + \cdots + B_m s^m$$

$$F(s) = F_0 + F_1 s + \cdots + F_{m+\mu} s^{m+\mu}$$

$$[A_0\ B_0\ \ A_1\ B_1\ \ A_2\ B_2\ \cdots\ A_m\ B_m]\, S_m$$

$$= [F_0\ F_1\ \cdots\ F_{\mu+m}] \triangleq F$$

$$S_m = \begin{bmatrix} D_0 & D_1 & \cdots & D_\mu & 0 \cdots 0 \\ N_0 & N_1 & \cdots & N_\mu & 0 \cdots 0 \\ 0 & D_0 & \cdots & & \\ 0 & N_0 & \cdots & & \\ & & D_0 & & \\ & & N_0 & \ddots & \end{bmatrix}$$

Let ν_i be the number of linearly independent i^{th} N-rows

$$\boxed{\nu} \triangleq \max\{\nu_1, \nu_2, \ldots, \nu_q\}$$

it is the row index of $G(s)$

$$\nu_1 + \nu_2 + \cdots \nu_q = n$$

Let $m_i \geq \nu - 1$ for $i = 1, \cdots, P$

$$H_c(s) = \operatorname{diag}(s^{\mu_1}, \dots, s^{\mu_p})$$
$$H_r(s) = \operatorname{diag}(s^{m_1}, \dots, s^{m_p})$$

Let $m = \max(m_1, m_2, \dots, m_p)$

$$\bar{F} = [F_0 \quad \dots \quad F_{\mu+m}]$$

For any $F(s)$ such that

$$\lim_{s \to \infty} H_r^{-1} F H_c^{-1} \triangleq F_h$$

is a nonsingular matrix, there exists a $p \times q$ proper compensator $A^{-1} \cdot B$ where $A(s)$ is row reduced with row degrees m_i

$$G_0 = N F^{-1} B$$
$$H_c = \operatorname{diag}(s^{\mu_1}, \dots s^{\mu_p})$$
$$H_r = \operatorname{diag}(s^{m_1}, \dots, s^{m_p})$$

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3. Example :

$$G(s) = \begin{bmatrix} \frac{1}{s^2} & \frac{1}{s} \\ 0 & \frac{1}{s} \end{bmatrix}$$

right coprime fractions:

$$G(s) = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{N} \underbrace{\begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix}^{-1}}_{D^{-1}}$$

$$D(s) = D_0 + D_1 s + D_2 s^2$$

$$\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{D_0} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{D_1} s + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{D_2} s^2$$

$$N(s) = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{N_0} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{N_1} s + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{N_2} s^2$$

$$F(s) = \begin{bmatrix} s^3 + 17s^2 + 17s + 15 & 0 \\ 0 & s^2 + 2s + 5 \end{bmatrix}$$

find A, B such that

$$AD + BN = F$$

$$S_m = \underbrace{\begin{bmatrix} D_0 & D_1 & D_2 & 0 \\ N_0 & N_1 & N_2 & 0 \\ 0 & D_0 & D_1 & D_2 \\ 0 & N_0 & N_1 & N_2 \end{bmatrix}}_{8} \quad 8$$

... let us check with matlab...

4. <u>before</u>, let us find a realization of $G(s)$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \qquad G(s) = \begin{bmatrix} \frac{1}{s^2} & \frac{1}{s} \\ 0 & \frac{1}{s} \end{bmatrix}$$

$\Rightarrow$ controllable canonical form is quite easy to construct, otherwise check the <u>structure theorem</u>.

$$A = \left[\begin{array}{cc:c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 \end{array}\right] \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G(s) = C(sI - A)^{-1}B$$

$\rightarrow$ check with Mathematica

- State feedback using the Lyapunov equation (recipe).

1°/ select an arbitrary F $(n \times n)$ matrix having the desired eigenvalues but none in common with A.

2°/ Select an arbitrary $\bar{K}$ matrix

$p \times n$ (2×3)

such that $(\bar{K}, F)$ is observable.

3°/ Compute the solution of the Lyapunov equation

$$AT - TF = B\bar{K}$$

lyap(A,B,C) in matlab solves

$$AX + XB + C = 0$$

$$\text{lyap}(A, -F, -B\bar{K}) \to T$$

4°/ $K = \bar{K} T^{-1}$